Representation of Fuzzy Quantum Posets of Types I, II

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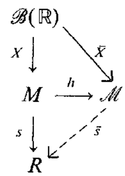
Let (Ω, M) be a fuzzy quantum poset of type I, II, or FQP of type I, II for short. For Boolean representations of fuzzy quantum spaces, by a representation of (Ω, M) we mean a quantum logic $\mathcal{M}$ (i.e., an orthocomplemented *σ-orthocomplete* orthomodular poset with a homomorphism $h: M \xrightarrow{onto} \mathcal{M}$ such that for any state s on M and any observable $\bar{X}$ on $\mathcal{M}$ there is a state $\bar{s}$ on $\mathcal{M}$ and observable X on M such that the following diagram commutes [where $\mathcal{B}(\mathbb{R})$ is a Borel σ-algebra of the real line $\mathbb{R}$]:



We prove that a representation of FQP of type I always exists and a representation of FQP of type II exists in some cases.

1. PRELIMINARIES

We recall that two fuzzy sets a, b are said to be *fuzzy orthogonal*, we write $a \perp_F b$, iff $a \cap b := \inf(a, b) \le 1/2$, and *orthogonal*, we write $a \perp b$, iff $a \le b^{\perp}$.

Let Ω be a nonempty set, M be a system of fuzzy sets, $M \subseteq [0, 1]^{\Omega}$, such that:

(i) $\mathbf{1}(\omega) = 1$ for any $\omega \in \Omega$, then $\mathbf{1} \in M$.

(ii) $a \in M$, then $a^{\perp} := 1 - a \in M$.

(iii) $\mathbf{1/2}(\omega) = 1/2$ for any $\omega \in \Omega$, then $\mathbf{1/2} \notin M$.

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A set $M \subseteq [0, 1]^{\Omega}$ satisfying conditions (i)–(iii) is said to be *an FQP of type I* (*of type II*) if it is closed with respect to a union of any sequence of mutually fuzzy orthogonal (mutually orthogonal) fuzzy sets, respectively, where by union we mean the union of Zadeh's connective. If M is closed with respect to a union of any sequence of fuzzy sets from M, then M is said to be a *fuzzy quantum space*, or FQS for short.

It is clear that $a \perp b$, then $a \perp_F b$ for any $a, b \in M$. So, an FQP of type I is an FQP of type II and and FQS is an FQP of type I (Dvurečenskij, n.d.; Long, 1992, n.d.; Riečan, 1988).

An observable X on (Ω, M) is a mapping $X: \mathscr{B}(\mathbb{R}) \to M$ such that:

(i) $X(E^c) = X(E)^{\perp}$ for any Borel set $E \in \mathscr{B}(\mathbb{R})$.

(ii) $X(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} X(E_i)$ for any sequence $\{E_i\}_{i=1}^{\infty} \in \mathscr{B}(\mathbb{R})$.

Denote by $\mathfrak{O}(M)$ the set of all observables on (Ω, M).

A mapping $m: M \to [0, 1]$ is said to be a state of type I, II on (Ω, M) if:

(i) $m(a) + m(a^{\perp}) = 1$ for $a \in M$.

(ii) $m(\bigcup_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ for any sequence of mutually fuzzy orthogonal, orthogonal, fuzzy sets $\{a_i\}_{i=1}^{\infty} \subseteq M$, respectively.

Denote by $\mathfrak{S}_{\mathrm{I}}(M)$, $\mathfrak{S}_{\mathrm{II}}(M)$ the sets of all states of types I, II on (Ω, M), respectively.

Proposition 1. Let (Ω, M) be an FQP of type I; then:

(i) $\mathfrak{S}_{\mathrm{I}}(M) \subseteq \mathfrak{S}_{\mathrm{II}}(M)$.

(ii) If (Ω, M) is an FQS, then $\mathfrak{S}_{\mathrm{I}}(M) = \mathfrak{S}_{\mathrm{II}}(M)$.

Now let (Ω, M) be an FQP of type I or FQP of type II such that

$$a \cap c \in M \qquad \text{for any } a, c \in M, \quad c \geq 1/2 \tag{2}$$

Consider a relation $\sim \subseteq M \times M$ defined by

$$a \sim b \qquad \text{if} \quad a \cap b^{\perp}, \quad a^{\perp} \cap b \leq 1/2$$

It is clear that (i) $a \sim a$ for any a from M; (ii) if $a \sim b$, then $a^{\perp} \sim b^{\perp}$; (iii) if $a \sim b$, then $b \sim a$, but $\sim$ is not transitive, in general. Let $\simeq$ be the transitive closure of $\sim$, i.e., the smallest quivalence relation on M containing $\sim$. It is obvious that $a \approx b$ iff there are $a_1, a_2, \ldots, a_n \in M$ such that $a \sim a_1, a_1 \sim a_2, \ldots, a_n \sim b$.

It can be proved that $a \approx b$ iff there is a $c \in M$, $c \geq 1/2$, such that

$$a \cap b^{\perp} \cap c, \qquad a^{\perp} b \cap c \leq 1/2$$

or equivalently

$$\{a \cap b^{\perp} > 1/2\} \bigcup \{a^{\perp} \cap b > 1/2\} \subseteq \{c = 1/2\}$$

where $\{a \cap b^{\perp} > 1/2\} := \{\omega \in \Omega; (a \cap b^{\perp})(\omega) > 1/2\}$, etc.

Note that if we consider $\Omega = [0, 1]$,

$$a(\omega) = \begin{cases} 0.7 & \text{if} \quad 0 \le \omega < 0.6 \\ 0.3 & \text{if} \quad 0.6 \le \omega \le 1 \end{cases}$$

$$b(\omega) = \begin{cases} 0.4 & \text{if} \quad 0 \le \omega < 0.6 \\ 0.6 & \text{if} \quad 0.8 \le \omega \le 1 \end{cases}$$

$c = a \cup a^{\perp}$; $d = b \cup b^{\perp}$; $e = d \cap a$; $f = d \cap a^{\perp}$; $g = a \cup f$; $h = b \cup e$; $i = e \cup d^{\perp}$; $k = e \cup e^{\perp}$; then

$M =$

$$\{\mathbf{0}, \mathbf{1}, a, b, c, d, e, f, g, h, i, j, k, a^{\perp}, b^{\perp}, c^{\perp}, d^{\perp}, e^{\perp}, f^{\perp}, g^{\perp}, h^{\perp}, i^{\perp}, j^{\perp}, k^{\perp},\}$$

is an FQP of type II with (2) but not a type I.

The following results can be proved in the same way as the proofs in Dvurečenskij and Long (1991).

Proposition 2. The transitive closure $\approx$ is a proper congruence relation in M.

Now, for any $a \in M$, we put $\bar{a} := \{b \in M; b \approx a\}$, and $\mathcal{M} := \{\bar{a}; a \in M\}$. In $\mathcal{M}$ we define a relation $\le$ via

$$\bar{a} \le \bar{b} \quad \text{iff} \quad \text{there is a } c \ge 1/2 \text{ and } a \cap b^{\perp} \cap c \le 1/2$$

and the mapping $\perp: \mathcal{M} \to \mathcal{M}$ defined via $\bar{a} \mapsto \bar{a}^{\perp}$, $a \in M$, then $\le$ and $\perp$ are well defined. It is easy to check that $\le$ is an order relation and $\perp$ is an orthocomplementation on $\mathcal{M}$.

Lemma 3. Let (Ω, M) be an FQP of type I or FQP of type II with (2):

(i) For any $a, c \in M$, $c \ge 1/2$; $a \approx a \cap c \cup c^{\perp}$.

(ii) For any $a, b \in M$, $\bar{a} \le \bar{b}$, then there are $a_1, b_1 \in M$ such that $a_1 \approx a$, $b_1 \approx b$ and $a_1 \le b_1$.

Proof. Part (i) is clear.

(ii) Since $\bar{a} \le \bar{b}$, there is $c \in M$, $c \ge 1/2$, such that $a \cap b^{\perp} \cap c \le 1/2$, then $a_1 := a \cap c \cup c^{\perp}$ and $b_1 := b \cap c \cup c^{\perp}$ satisfy the conditions of the theorem.

Theorem 4. Let (Ω, M) be an FQP of type I or FQP of type II with (2); then $\mathcal{M}$ equipped with an order relation $\le$ and an orthocomplementation $\perp$ is a quantum logic with the least element $\bar{\mathbf{0}}$ and the greatest element $\bar{\mathbf{1}}$ and $h: M \to \mathcal{M}$ defined via $a \mapsto \bar{a}$ is a σ-homomorphism from M onto $\mathcal{M}$—i.e., $h(a^{\perp}) = h(a)^{\perp}$ and $h(\bigcup_{i=1}^{\infty} a_i) = \bigcup_{i=1}^{\infty} h(a_i)$ for any sequence of mutually fuzzy orthogonal, orthogonal, fuzzy sets, respectively.

Let (Ω, M) be an FQP of type II; we put

$$\mathscr{K}(M) = \{A \subseteq \Omega;\ \exists a \in M;\ \{a > 1/2\} \subseteq A \subseteq \{a \geq 1/2\}\} \tag{3}$$

$$\mathscr{I}(M) = \{A \subseteq \Omega;\ \exists a \in M;\ A \subseteq \{a = 1/2\}\}$$

There are two constructions of representations of FQP (Dvurečenskij, n.d.; Dvurečenskij and Long, 1991). The following proposition shows that they are equivalent.

Proposition 5. Let (Ω, M) be an FQP of type I or FQP for type II with (2); then:

(i) $\mathscr{K}(M)$ is a q-σ-algebra and $\mathscr{I}(M)$ is an σ-ideal of $\mathscr{K}(M)$ [i.e., $\mathscr{K}(M)$ is a system of subsets of Ω which is closed with respect to complementation and countable union of mutually disjoint subsets, $\mathscr{I}(M)$ is a nonempty subset of $\mathscr{K}(M)$ closed with respect to countable union of mutually disjoint subsets, and if $A \in \mathscr{K}(M)$, $B \in \mathscr{I}(M)$, $A \subseteq B$, then $A \in \mathscr{I}(M)$].

(ii) Consider a mapping $g\colon \mathscr{K}(M) \to \mathscr{M}$, defined via $A \mapsto \bar{a}$, where A, a satisfy (3); then g defines well a σ-homomorphism from $\mathscr{K}(M)$ onto $\mathscr{M}$ and $g^{-1}(\bar{0}) = \mathscr{I}(M)$. Moreover, we consider on $\mathscr{K}(M)$ a relation θ: for any $A, B \in \mathscr{K}(M)$, $A\theta B$ iff $A\backslash B$, $B\backslash A \in \mathscr{I}(M)$, then θ is a congruence relation on $\mathscr{K}(M)$. Put, for any $A \in \mathscr{K}(M)$,

$$\bar{A} := \{B \in \mathscr{K}(M);\ B\theta A\}, \qquad \mathscr{K}(M)/\theta := \{\bar{A};\ A \in \mathscr{K}(M)\}$$

Define

$$\bar{A}^{\perp} := \overline{A^c} \quad \text{and} \quad \bar{A} \leq \bar{B} \quad \text{iff} \quad A\backslash B \in \mathscr{I}(M)$$

Then $\perp$, $\leq$ is well defined, an orthocomplementation, and an order relation on $\mathscr{K}(M)/\theta$ such that $\mathscr{K}(M)/\theta$ with $\perp$, $\leq$ is a quantum logic and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{K}(M) & \xrightarrow{\ g\ } & \mathscr{M} \\ {\scriptstyle Pr}\uparrow & \nearrow{\scriptstyle \cong} & \\ \mathscr{K}(M)/\theta & & \end{array}$$

where Pr is a projection.

2. A REPRESENTATION OF TYPES I, II FQP

Theorem 6. Let (Ω, M) be an FQP of type I or FQP of type II with (2); then for any observable $\bar{X}$ on $\mathscr{M}$ there is an observable X on M such that $\bar{X} = h \circ X$, where h, $\mathscr{M}$ from Theorem 4.

Proof. Let $\bar{X}$ be an observable on $\mathcal{M}$ and $\mathbb{Q}$ be the set of rational numbers. Consider $\overline{a_r} := \bar{X}((-\infty, r))$, $r \in \mathbb{Q}$; then $\overline{a_r} \leq \overline{a_s}$ if $r \leq s$. Owing to Lemma 3, we can set up a sequence b_r, $r \in \mathbb{Q}$, such that $h(b_r) = \overline{a_r}$ and $b_r \leq b_s$ if $r \leq s$. Due to Theorem 1.4 of Varadarajan (1968) and Theorem 4.5 of Long (n.d.) there is an observable X on M such that $X((-\infty, r)) = b_r$. Therefore, $\bar{X} = h \circ X$.

Theorem 7. Let (Ω, M) be an FQP of type I; then for any $m \in \mathfrak{S}_{\mathrm{I}}(M)$, $\bar{m}: \mathcal{M} \to [0, 1]$ defined by $\bar{m}(\bar{a}) = m(a)$, $a \in M$, is a state on M. Conversely, for any $s \in \mathfrak{S}_{\mathrm{I}}(M)$ there is a state $m \in \mathfrak{S}_{\mathrm{I}}(M)$ such that $\bar{m} = s$.

Proof. The theorem can be proved in the same way as the proofs in Dvurečenskij (n.d.).

Corollary 8. Let (Ω, M) be an FQP of type II with (2) such that for any $m \in \mathfrak{S}_{\mathrm{II}}(M)$; for any $a, b \in M$, $a \cap b^{\perp} \leq 1/2$, $a^{\perp} \cap b \leq 1/2$, imply $m(a) = m(b)$, then for any $m \in \mathfrak{S}_{\mathrm{II}}(M)$, $\bar{m}: \mathcal{M} \to [0, 1]$ defined by $\bar{m}(\bar{a}) = m(a)$, $a \in M$, is a state on M.

Conversely, for any $s \in \mathfrak{S}(\mathcal{M})$ there is a state $m \in \mathfrak{S}_2(M)$ such that $\bar{m} = s$.

Theorem 9. Let (Ω, M) be an FQP of type I or FQP of type II satisfying the conditions of Corollary 8; then $\mathcal{M}$ with σ-homomorphism h from Theorem 4 is a representation of M.

3. CONCLUSION

We have solved the problem of representation of an FQP of type I and some kinds of FQP of type II. We can also point out that there is an FQP of type II which has no representation. Finally, natural questions arise: Is any quantum logic a representation of some FQP? We note that in Varadarajan (1968, Theorem 2.2.5) it is proved that every logic is a surjective homomorphic image of a concrete logic. But the conditions of a representation in our sense are not satisfied in general.

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REFERENCES

Dvurečenskij, A. (n.d.). Remarks on representation of fuzzy quantum posets, submitted.

Dvurečenskij, A., and Long, L. B. (1991). On representations of fuzzy quantum posets, *Acta Mathematicae Universitas Comenianae*, **58–59**, 239–245.

Long, L. B. (1992). A new approach to representation of observables on fuzzy quantum posets, *Applications of Mathematics*, **37**(5), 357–368.

Long, L. B. (n.d.). Compatibility and representation of observable on type I, II, III of fuzzy quantum posets, *Fuzzy Sets and Systems*, submitted.

Navara, M. (n.d.). Boolean representation of fuzzy quantum space, to appear.

Riečan, B. (1988). A new approach to some notions of statistical quantum mechanics, *Busefal*, **35**, 4–6.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Van Nostrand, New York.

Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.